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Facts: If $\mathcal{U}$ is a universal set

(a) $\overline{\mathcal{U}} = \emptyset$

(b) $\overline{\emptyset} = \mathcal{U}$

1.8: Indexed Sets

Consider the sequence

$$\{a_1, a_2, a_3, \cdots\}$$

We could say that it is *indexed* by $\mathbb{N}$ since we could write it as $\{a_n\}_{n\in\mathbb{N}}$, that is, to every element of $\mathbb{N}$ there corresponds an entry in the sequence.

We could move this idea to sets to create the idea of *indexed sets*.

Suppose we have a list of 30 sets, too many to give one letter to each set. We could instead list them as:

$$A_1, A_2, A_3, \cdots, A_{29}, A_{30}$$

We could write a collection of these sets as

$$\begin{aligned} \mathcal{A} &= \{A_n \mid n \in \{1, \cdots, 30\}\} \\ &= \{A_n \mid n \in \mathbb{N}, 1 \le n \le 30\} \quad \text{etc.} \end{aligned}$$

(18)

When we have a collection of sets like this, we can form some new sets.

<u>Def</u>: Suppose we have a collection of sets $\mathcal{A} = \{A_1, A_2, \cdots, A_n\}$. We have

① $A_1 \cup A_2 \cup \cdots \cup A_n = \{x \mid x \in A_i \text{ for at least one set } A_i, 1 \le i \le n\}$
$$= \bigcup_{i=1}^{n} A_i = \bigcup_{i \in \{1,2,\cdots,n\}} A_i = \bigcup \mathcal{A}$$

② $A_1 \cap A_2 \cap \cdots \cap A_n = \{x \mid x \in A_i \text{ for all sets } A_i, 1 \le i \le n\}$
$$= \bigcap_{i=1}^{n} A_i = \bigcap_{i \in \{1,\cdots,n\}} A_i = \bigcap \mathcal{A}$$

<u>Ex</u>: Let $A_1 = \{0,2,4,8,10,12,14,16,18,20,22,24\}$
$A_2 = \{0,3,6,9,12,15,18,21,24\}$
$A_3 = \{0,4,8,12,16,20,24\}$

$$\bigcap_{i=1}^{3} A_i = \{0,12,24\}$$

$$\bigcup_{i=1}^{3} A_i = \{0,2,3,4,6,8,9,10,12,14,15,16,18,20,21,22,24\}$$

We could also have infinite collections of sets too. If we have $\mathcal{A} = \{A_1, A_2, A_3, \ldots\}$, then we could write $\mathcal{A} = \{A_i \mid i \in \mathbb{N}\}$. Thus the intersection and unions could be written as

$$A_1 \cup A_2 \cup A_3 \cup \cdots = \{x \mid x \in A_i \text{ for some set } A_i,\ i \geq 1\}$$
$$= \bigcup_{i=1}^{\infty} A_i = \bigcup_{i \in \mathbb{N}} A_i = \bigcup \mathcal{A} = \bigcup_{i \geq 1} A_i$$

$$A_1 \cap A_2 \cap A_3 \cap \cdots = \{x \mid x \in A_i \text{ for all sets } A_i,\ i \geq 1\}$$
$$= \bigcap_{i=1}^{\infty} A_i = \bigcap_{i \in \mathbb{N}} A_i = \bigcap \mathcal{A} = \bigcap_{i \geq 1} A_i$$

Ex: Let $A_n = \{n^p \mid p \in \mathbb{N}_0\}$ (Reminder $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}$)

$$\bigcup_{n \geq 2} A_n = \{1, 2, 3, 4, \ldots\} = \mathbb{N}.$$

$$\bigcap_{n \geq 2} A_n = \{1\}$$

So far, we've only seen collections indexed by $\mathbb{N}$, but we could have things more general as well.

Let I be any set, and for every $i \in I$ associate a set A_i. Then we call I an <u>index set</u>.

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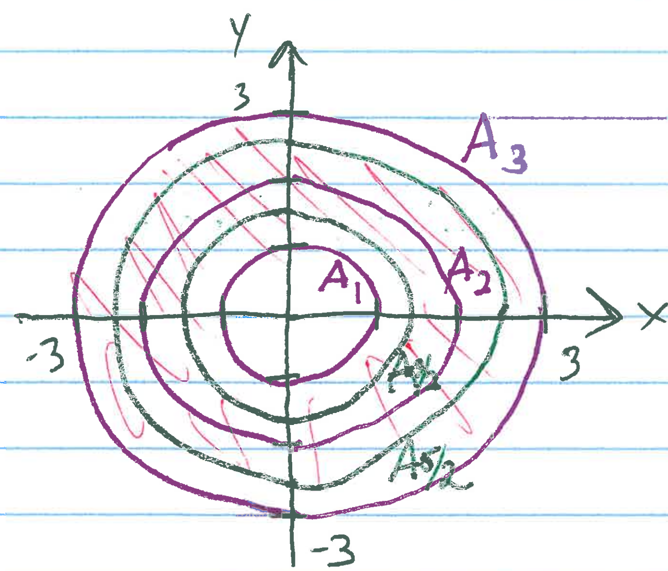
We could define similarly to before

$\bigcap_{i \in I} A_i$ and $\bigcup_{i \in I} A_i$.

Ex: Let $A_r = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = r^2\}$

Then A_r is the subset of $\mathbb{R}^2$ which is the circle of radius r.

$\bigcup_{r \in [1,3]} A_r$ = the annulus with outer radius 3 and inner radius 1.



$\bigcap_{r \in [1,3]} A_r = \emptyset$ since none of the circles intersect.

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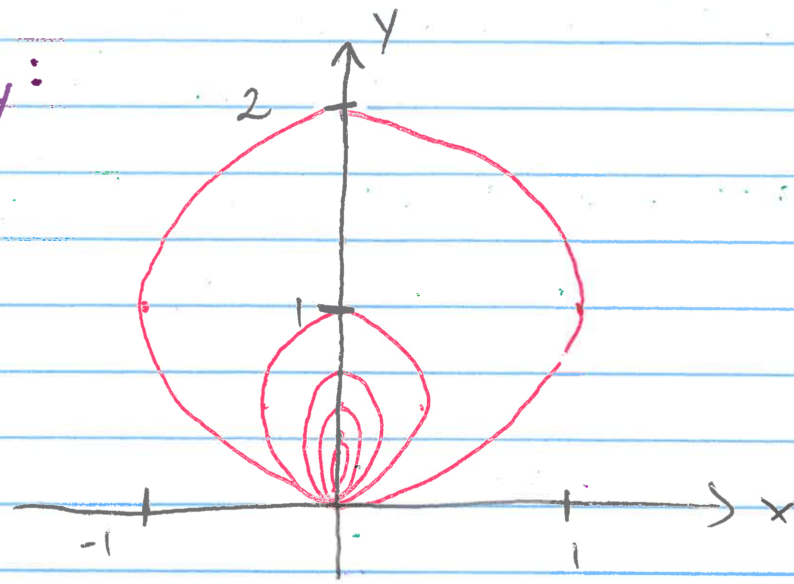
Ex: A peculiar example, useful in topology, called the Hawaiian earring is given by

$$H = \bigcup_{n=1}^{\infty} C_n$$

where $C_n = \{(x,y) \in \mathbb{R}^2 \mid x^2 + (y - \frac{1}{n})^2 = \frac{1}{n^2}\}$

Visually:



What is $\bigcap_{n=1}^{\infty} C_n$? Just $\{(0,0)\}$.

1.10: Russell's Paradox

Most sets we think of have the property that $A \notin A$, e.g., $\mathbb{Z} \notin \mathbb{Z}$, $\{1,2,3\} \notin \{1,2,3\}$.
But are there sets B with $B \in B$?
Yes, let $B = \{\{\{\{\cdots\}\}\}\}$, an infinite nesting of braces.

So, what of the set

$$P = \{X \mid X \text{ is a set and } X \notin X\}\ ?$$

Is $P \in P$ or is $P \notin P$?

If $P \in P$, then by definition of P, $P \notin P$, which is a contradiction.

But, if $P \notin P$, then since P is a set, by definition of P, $P \in P$.

This is Russell's paradox!

This shows the limit of set theory.

The Zermelo-Frankel axioms lay the foundation of set theory, but we often also include the so-called axiom of choice.